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Chiral symmetry breaking

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References: Chiral symmetry breaking is discussed in Cheng & Li sections 5.4 and 5.5, but using a rather old-fashioned algebraic approach. Peskin & Schroeder discuss chiral symmetry breaking on pages 667 – 670.

Now we're ready to see how some of these ideas of symmetries and symmetry breaking are realized by the strong interactions. But first, some terminology. If one can decompose

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

where $\mathcal{L}_0$ is invariant under a symmetry and $\mathcal{L}_1$ is non-invariant but can be treated as a perturbation, then one has "explicit symmetry breaking" by a term in the Lagrangian. This is to be contrasted with "spontaneous symmetry breaking," where the Lagrangian is invariant but the ground state is not. Incidentally, one can have both spontaneous and explicit symmetry breaking, if $\mathcal{L}_0$ by itself breaks the symmetry spontaneously while $\mathcal{L}_1$ breaks it explicitly.

Let's return to the quark model of section 3.1. For the time being we'll ignore quark masses. With three flavors of quarks assembled into

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

we guessed that the strong interaction Lagrangian looked like

$$\mathcal{L}_{\text{strong}} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \cdots$$

As discussed in section 5.1 the quark kinetic terms have an $SU(3)$ symmetry

$\psi \to U\psi$. Assuming this symmetry extends to all of $\mathcal{L}_{\text{strong}}$ the corresponding conserved currents are

$$j^{\mu\, a} = \bar{\psi}\gamma^\mu T^a \psi$$

where the generators T^a are 3×3 traceless Hermitian matrices.

In fact the quark kinetic terms have a larger symmetry group. To make this maifest we need to decompose the Dirac spinors u, d, s into their left- and right-handed chiral components. The actual calculation is identical to what we did for QED in section 4.1. The result is

$$\mathcal{L}_{\text{strong}} = \bar{\psi_L} i\gamma^\mu \partial_\mu \psi_L + \bar{\psi_R} i\gamma^\mu \partial_\mu \psi_R + \cdots$$

Here

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \qquad \psi_R = \frac{1}{2}(1+\gamma^5)\psi$$

and

$$\bar{\psi_L} \equiv (\psi_L)^\dagger \gamma^0 \qquad \bar{\psi_R} \equiv (\psi_R)^\dagger \gamma^0 \,.$$

The easiest way to get this result is to note that the projection operators

$$P_L = \frac{1}{2}(1-\gamma^5) \qquad P_R = \frac{1}{2}(1+\gamma^5)$$

satisfy

$$P_L^2 = P_L \qquad P_R^2 = P_R \qquad P_L + P_R = \mathbb{1}\,.$$

In any case this chiral decomposition makes it clear that the quark kinetic terms actually have an $SU(3)_L \times SU(3)_R$ symmetry that acts independently on the left- and right-handed chiral components.†

$$\psi_L \to L\psi_L \qquad \psi_R \to R\psi_R \qquad L, R \in SU(3) \tag{6.1}$$

It's easy to work out the corresponding conserved currents; they're just what we had above except they only involve one of the chiral components:

$$j_L^{\mu\, a} = \bar{\psi_L}\gamma^\mu T^a \psi_L = \bar{\psi}\gamma^\mu T^a \frac{1}{2}(1-\gamma^5)\psi$$
$$j_R^{\mu\, a} = \bar{\psi_R}\gamma^\mu T^a \psi_R = \bar{\psi}\gamma^\mu T^a \frac{1}{2}(1+\gamma^5)\psi$$

† The full symmetry is $U(3)_L \times U(3)_R$. As we've seen the extra vector-like $U(1)$ corresponds to conservation of baryon number. The fate of the extra axial $U(1)$ is a fascinating story I can't get in to now.

It's often convenient to work in terms of the "vector" and "axial-vector" combinations

$$\begin{aligned} j_V^{\mu\,a} &= j_L^{\mu\,a} + j_R^{\mu\,a} = \bar{\psi}\gamma^\mu T^a \psi \\ j_A^{\mu\,a} &= -j_L^{\mu\,a} + j_R^{\mu\,a} = \bar{\psi}\gamma^\mu \gamma^5 T^a \psi \end{aligned}$$

The question is what to make of this larger symmetry group. As we've seen the vector current corresponds to Gell-Mann's flavor $SU(3)$. But what about the axial current?

The simplest possibility would be for $SU(3)_A$ to be a manifest symmetry of the particle spectrum.† We can rule this out right away. The axial charges

$$Q_A^a = \int d^3x\, j_A^{0\,a}$$

are odd under parity (see Peskin & Schroeder p. 65), so they change the parity of any state they act on. If $SU(3)_A$ were a manifest symmetry there would have to be scalar (as opposed to pseudoscalar) particles with the same mass as the pions.

Another possibility is for $SU(3)_A$ to be explicitly broken by $\mathcal{L}_{\text{strong}}$: after all we've only been looking at the quark kinetic terms. I can't say anything against this possibility, except that we might as well assume $SU(3)_A$ is a valid symmetry and see where that assumption leads.

So we're left with the idea that $SU(3)_A$ is a valid symmetry of the strong interaction Lagrangian, but is spontaneously broken by a choice of ground state. What order parameter could signal symmetry breaking? It's a bit subtle, but suppose the fermion billinear $\psi\bar{\psi}$ aquires an expectation value:

$$\langle 0|\psi\bar{\psi}|0\rangle = \mu^3 \mathbb{1}\,.$$

Here μ is a constant with dimensions of mass, and $\mathbb{1}$ is the identity matrix both in flavor space and spinor space. In terms of chiral spinors‡ $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ this is equivalent to

$$\langle \psi_L \psi_R^\dagger \rangle = \mu^3 \mathbb{1} \qquad \langle \psi_R \psi_L^\dagger \rangle = \mu^3 \mathbb{1} \qquad \langle \psi_L \psi_L^\dagger \rangle = \langle \psi_R \psi_R^\dagger \rangle = 0\,. \tag{6.2}$$

What's nice is that this expectation value

- is invariant under Lorentz transformations (check!)

† Picky, picky: the symmetry group is really $SU(3)_L \times SU(3)_R$. The linear combination $R - L$ that appears in the axial current doesn't generate a group, since two axial charges commute to give a vector charge. I'll call the axial symmetries $SU(3)_A$ anyways.

‡ I'm switching notation a bit. Previously ψ_L and ψ_R were 4-component objects, Dirac spinors two of whose components vanished. Now ψ_L and ψ_R denote 2-component objects.

- is invariant under $SU(3)_V$ transformations $\psi_L \to U\psi_L$, $\psi_R \to U\psi_R$
- completely breaks the $SU(3)_A$ symmetry

That is, in (6.1) one needs to set $L = R$ in order to preserve the expectation value (6.2).

So the claim is that strong-coupling effects in QCD cause $q\bar{q}$ pairs to condense out of the trivial (perturbative) vacuum; the expectation value (6.2) is supposed to be generated dynamically by the strong interactions.

In fact, the expectation value can be a bit more general. Whenever a continuous symmetry is spontaneously broken there should be a manifold of inequivalent vacua. We can find this space of vacua just by applying $SU(3)_L \times SU(3)_R$ transformations to the vev (6.2). The result is

$$\langle \psi_L \psi_R^\dagger \rangle = \mu^3 U \qquad \langle \psi_R \psi_L^\dagger \rangle = \mu^3 U^\dagger \qquad \langle \psi_L \psi_L^\dagger \rangle = \langle \psi_R \psi_R^\dagger \rangle = 0$$

where $U = e^{i\lambda^a T^a}$ is an $SU(3)$ matrix. In terms of Dirac spinors this is equivalent to

$$\langle 0 | \psi \bar{\psi} | 0 \rangle = \mu^3 e^{-i\lambda^a T^a \gamma^5} \,. \tag{6.3}$$

If this is right, the space of vacua of QCD is labeled by an $SU(3)$ matrix U. We'd expect to have $\dim SU(3) = 8$ massless Goldstone bosons that can be described by a field $U(t, \mathbf{x})$. If we're at very low energies then the dynamics of QCD reduces to an effective theory of the Goldstone bosons. What could the action be? As we'll discuss in more detail in the next chapter, there's a unique candidate with at most two derivatives: the non-linear σ-model action from the last homework!

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} f^2 \text{Tr} \left(\partial_\mu U^\dagger \partial^\mu U \right)$$

This action provides a complete description of the low-energy dynamics of QCD with three massless quarks.

In the real world various effects (quark mass terms, electromagnetism, ...) explicitly break $SU(3)_A$. This turns out to give the would-be Goldstone bosons a small mass. But we'd still expect to have eight anomalously light scalar particles – the meson octet π, K, η!

Finally we can understand the origins of $SU(3)_{\text{flavor}}$ symmetry and why the meson octet is so light. Current estimates are that the "chiral condensate" has a value

$$\langle \bar{u}u \rangle \approx \langle \bar{d}d \rangle \approx \langle \bar{s}s \rangle \approx (260\,\text{MeV})^3$$

This is large compared to the light quark masses

$$m_u \approx 3\,\mathrm{MeV} \qquad m_d \approx 6\,\mathrm{MeV} \qquad m_s \approx 120\,\mathrm{MeV}$$

but small compared to the heavy quark masses

$$m_c \approx 1.2\,\mathrm{GeV} \qquad m_b \approx 4.2\,\mathrm{GeV} \qquad m_t \approx 175\,\mathrm{GeV}$$